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Some remarks on a Telgarsky's conjecture concerning products of paracompact spaces

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ABSTRACT

R. Telgarsky conjectured that if X is a paracompact space then the product $X \times Y$ is paracompact for every paracompact space Y if and only if the first player of the $G(DC, X)$ game, introduced by R. Telgarsky, see [R. Telgarsky, Spaces defined by topological games, Fund. Math. 88 (1975) 193–223], has a winning strategy. The paper contains some results supporting this conjecture.

Let us denote by $\mathcal{P}$ the class of all spaces whose Cartesian product with every paracompact space is paracompact. The general question is to characterize the class $\mathcal{P}$ and to verify whether the class is closed with respect to closed mappings and X^ω is paracompact provided that X belongs to $\mathcal{P}$. We adopt the topological terminology from [3] and set-theoretical from [4]. By a P-space we mean a space whose topology is closed with respect to countable intersections. In the sequel ω and ω_1 stand for the first infinite ordinal number and the first uncountable ordinal number, respectively. The symbol Lim stands for the limit countable ordinal numbers. The ordinal numbers are denoted by Greek letters, the set of natural numbers by N and the unit interval by I. All spaces considered in the paper are regular and mappings are continuous. If A is a set then the symbol $|A|$ stands for the cardinality of A. If $Z = Y^\lambda$, $S \subset \lambda$, then p_S stands for the projection from Z onto Y^S; in particular, for $\alpha < \lambda$, p_α is the projection from Z onto Y^α. The hedgehog of spininess τ, where τ is a cardinal number, is denoted by $J(\tau)$.

In the sequel if X is a topological space and F is a subspace of X then X_F stands for the set X with the topology generated by the sets of the form $\{x\}$, where $x \in X \setminus F$ or U, where U is an open subset of X. The space R_Q, where R is the space of real numbers and Q the subspace of rational numbers, is defined according to this definition.

There are many references which are not mentioned in the text but they are related in a broad sense to the subject of the paper and therefore they are included in it.

For a topological space X we shall use notion of the $G(DC, X)$ game introduced by R. Telgarsky in [9]. Let us recall that the $G(DC, X)$ is a game with two players. The first player chooses odd numbered closed subsets F_n of X and the second player chooses even numbered closed subsets F_n of X such that:

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(i) each set selected by the first player is a discrete union of compact sets,
(ii) $F_{2k+1} \cap F_{2k+2} = \emptyset$, for $k \in N \cup \{0\}$,
(iii) $F_{2k+1} \subset F_{2k}$, for $k \in N$,
(iv) $F_{2(k+1)} \subset F_{2k}$, for $k \in N$.

We say that the first player wins the game if $\bigcap\{F_{2k}\colon k \in N\} = \emptyset$. The second player wins the game if $\bigcap\{F_{2k}\colon k \in N\} \neq \emptyset$. We say that a finite sequence $(F_1, F_2, \ldots, F_i)$ of closed subsets of X is admissible if the sets of it satisfy the rules of the $G(DC, X)$ game. We say that the first player has a winning strategy if there is a function s assigning to each admissible sequence $(P_1, P_2, \ldots, P_{2k})$ a discrete union of compact sets $s(P_1, \ldots, P_{2k}) = P_{2k+1}$, which is a subset of P_{2k}, for $k \in N$, such that for each sequence $(F_n)_{n=1}^{\infty}$, where $(F_1, F_2, \ldots, F_i)$ is admissible for each $i \in N$ and $s(F_1, F_2, \ldots, F_{2k}) = F_{2k+1}$, for $k \in N$, we have $\bigcap\{F_{2k}\colon k \in N\} = \emptyset$.

We say that a space X is σC-scattered (σDC-scattered), respectively, if for every closed subset F of X there is $x \in F$ and a closed neighborhood of x in F which is σ-compact (countable union of sets each of which is a discrete union of compact sets). We say that a space X is $\sigma_p DC$-scattered (G-scattered), respectively, if for every closed subset F of X there is $x \in F$ and a closed neighborhood H of x in F such that H is a countable pairwise disjoint union of sets each of which is a discrete union of compact sets (the first player in the $G(DC, H)$ game has a winning strategy).

The known results lead to the following conjectures, which I attribute to R. Telgarsky:

Conjecture 1. *A paracompact space X belongs to $\mathcal{P}$ if and only if the first player in the $G(DC, X)$ has a winning strategy.*

The implication that if X is a paracompact space and the first player in the game $G(DC, X)$ has a winning strategy then X belongs to $\mathcal{P}$ was proved by R. Telgarsky, see [9].

Conjecture 2. *If X is a paracompact space and the second player in the $G(DC, X)$ game has a winning strategy then X does not belong to $\mathcal{P}$.*

Remark 1. The positive answer to the first conjecture implies the positive answer to the second conjecture.

In the paper we make extensive use of the celebrated Michael type construction, see [5] and of the construction from [1].

The aim of this note is to present the following results:

Theorem 1. *If X is a paracompact $\sigma_p DC$-scattered space then $X \in \mathcal{P}$ if and only if the first player in the $G(DC, X)$ game has a winning strategy.*

Theorem 2. *If X is a σDC-scattered paracompact space which can be embedded in $(J(\tau))^{\omega_1}$, for a certain cardinal number τ then $X \in \mathcal{P}$ if and only if the first player of the $G(DC, X)$ game has a winning strategy.*

Theorem 3. *If X is a paracompact G-scattered space which can be embedded in $(J(\tau))^{\omega_1}$, for a certain cardinal number τ, in such a way that the projections of X onto initial countably many coordinates are closed, then $X \in \mathcal{P}$ if and only if the first player of the $G(DC, X)$ game has a winning strategy.*

In [1] is presented a theorem from which it follows that Conjecture 1 has a positive answer if X is an ω_1-metrizable space. For related results see also [2].

In the sequel we shall need some lemmas.

The first fact we need is a special case of Theorem 1.1 from [6]. We give a proof of it for the sake of completeness.

Lemma 1. *If $f : X \to X_1$ is a closed mapping from a paracompact space X onto a space X_1 satisfying the first axiom of countability then there is a closed subset F of X such that the restriction $f_1 = f|F$ is perfect and $f(X) = f_1(F)$.*

Proof. Let $y \in f(X)$ and $\{U_n\colon n \in N\}$ be a base at y in X_1. Put $U_y = \operatorname{int}(f^{-1}(y)$ and $F_y = f^{-1}(y) \setminus U_y$ if $f^{-1}(y) \setminus U_y \neq \emptyset$ and $F_y = \{x_y\}$ if $f^{-1}(y) \subset U_y$, where x_y is an arbitrary point of $f^{-1}(y)$. Then F_y is compact. Suppose not. Then there is an infinite and discrete set $\{x_n\colon n \in N\} \subset F_y$ and open sets G_n in X such that $\{G_n\colon n \in N\}$ is discrete in X, $x_n \in G_n \subset f^{-1}(U_n)$ and $G_n \setminus f^{-1}(y) \neq \emptyset$. Choose $z_n \in G_n \setminus f^{-1}(y)$ for each $n \in N$. Note that $\{z_n\colon n \in N\}$ is discrete in X and $(f(z_n))_{n=1}^{\infty}$ converges to y, contradicting the assumption that f is closed. Put $F = \bigcup\{F_y\colon y \in f(X)\}$. Then F is closed, $f_1 = f|F$ is perfect and $f_1(F) = f(X)$. $\square$

Lemma 2. *If $X \in \mathcal{P}$ and $f : X \to X_1$, where X_1 is a metrizable and f is a continuous and closed mapping from X onto X_1, then X_1 is a countable union of sets each of which is a discrete union of compact sets.*

Proof. By Lemma 1 there is a closed subset F of X such that the restriction $f_1 = f|F$ is perfect and $f(X) = f_1(F)$. K. Morita [7] observed that an immediate consequence of the E. Michael result, see [5] and the A.H. Stone result, see Theorem 2, [8], is the fact that a metrizable space belongs to $\mathcal{P}$ if and if it is a countable union of sets each of which is a discrete union of compact sets. The conclusion of the lemma follows from this result and from the fact that the class $\mathcal{P}$ is invariant with respect to perfect mappings. $\square$

The next lemma belongs to folklore.

Lemma 3. *A space X is σC-scattered if and only if there is an ordinal number α and the sequence $X^{(0)}_{\sigma C} = X \supset X^{(1)}_{\sigma C} \supset \cdots \supset X^{(\alpha)}_{\sigma C} = \emptyset$ such that if $\beta = \gamma + 1$ then $X^{(\beta)}_{\sigma C} = X^{(\gamma)}_{\sigma C} \setminus \{x \in X^{(\gamma)}_{\sigma C}\colon$ there is $U_x \subset X^{(\gamma)}_{\sigma C}\}$, where U_x is a closed neighborhood of x in $X^{(\gamma)}_{\sigma C}$ which is σ-compact. If β is a limit ordinal number then $X^{(\beta)}_{\sigma C} = \bigcap\{X^{(\lambda)}_{\sigma C}\colon \lambda < \beta\}$.*

If X is a σC-scattered space then put $ht_{\sigma C}(X) = \inf\{\gamma\colon X^{(\gamma)}_{\sigma C} = \emptyset\}$. A similar results hold for σDC-scattered spaces, $\sigma_p DC$-scattered spaces and G-scattered spaces. A suitable notions are denoted by the following symbols $X^{(\alpha)}_{\sigma DC}$, $X^{(\alpha)}_{\sigma_p DC}$, $X^{(\alpha)}_{G}$ and $ht_{\sigma DC}(X)$, $ht_{\sigma_p DC}(X)$ and $ht_G(X)$ for a given ordinal number α.

Lemma 4. *If X is a paracompact space, F is a closed subset of X and Y is a space such that the product $X \times Y$ is paracompact, then the product $X_F \times Y$ is paracompact.*

Proof. Let $\mathcal{U}$ be an open cover of $X_F \times Y$. Then we can find an open family $\mathcal{U}_1$ in $X \times Y$ refining $\mathcal{U}$ and covering $F \times Y$. Since $X \times Y$ is paracompact, we can construct a closed locally finite family $\mathcal{F}$ in $X \times Y$ covering $F \times Y$ and refining $\mathcal{U}_1$. Using paracompactness of $X \times Y$ we can extend $\mathcal{F}$ to a locally finite open family $\mathcal{H}$ in $X \times Y$ refining $\mathcal{U}_1$. Let G be a closed set in $X \times Y$ such that $F \times Y \subset \operatorname{int} G \subset G \subset \bigcup \mathcal{H}$. Then $(X_F \times Y) \setminus \operatorname{int} G$ is paracompact, as a discrete union of paracompact spaces. If $\mathcal{H}_1$ is an open locally finite family in $(X_F \times Y) \setminus \operatorname{int} G$ refining $\mathcal{U}$ and covering $(X_F \times Y) \setminus \operatorname{int} G$, then $\mathcal{H} \cup \mathcal{H}_2$, where $\mathcal{H}_2 = \{H \setminus G\colon H \in \mathcal{H}_1\}$, is a locally finite open refinement of $\mathcal{U}$ covering $X_F \times Y$. $\square$

The next fact was proved by R. Telgarsky, see [9].

Lemma 5. *In the definition of the sets chosen by the first player of the $G(DC, X)$ game we can replace the word "discrete" by "locally finite".*

Lemma 6. *If X is a space such that $X = \bigcup\{F_{nt}\colon n \in N,\ t \in T_n\}$, where the family $\{F_{nt}\colon t \in T_n\}$ is closed, discrete and the first player in the $G(DC, F_{nt})$ game has a winning strategy, for $n \in N$ and $t \in T_n$, then the first player in the $G(DC, X)$ game has a winning strategy.*

The proof is straightforward.

Lemma 7. *If X is a paracompact space and each $x \in X$ has a closed neighborhood U_x such that the first player in the $G(DC, U_x)$ has a winning strategy, then the first player in the $G(DC, X)$ has a winning strategy.*

The proof follows from Lemma 6.

Lemma 8. *Let X be a space such that for every paracompact space Y the product $X \times Y$ is paracompact. If X is a subspace of $J(\lambda)^\lambda$, for a certain cardinal number λ, then for every closed subset F of X and a countable subset S of λ there is a σ-closed subset Z of X such that*

$$p_S^{-1}\big(p_S(X) \setminus p_S(F)\big) \subset Z \subset X \setminus F.$$

Proof. Put $Y = p_S(X) \setminus p_S(F)$ with the topology of a subspace of $(J(\lambda))^S$, $K_1 = F \times Y$ and $K_2 = \bigcup\{p_S^{-1}(y) \times \{y\}\colon y \in Y\}$. Then K_1 and K_2 are disjoint and closed subsets of $X \times Y$. By normality of $X \times Y$, there are disjoint open subsets $U_1 \supset K_1$ and $U_2 \supset K_2$ of $X \times Y$. Let $\mathcal{B} = \bigcup\{\mathcal{B}_n\colon n \in N\}$ be a base of $p_S(X)$ such that $\mathcal{B}_n$, for $n \in N$, is a discrete family in $p_S(X)$. Put $X(B) = \{x \in p_S^{-1}(Y)\colon \{x\} \times \{p_S(x)\} \subset \{x\} \times B \cap Y \subset U_2\}$, for $B \in \mathcal{B}$. Since $U_1 \cap U_2 = \emptyset$, we infer that $\overline{X(B)} \cap F = \emptyset$. The family $\mathcal{B}_n$ is discrete in $p_S(X)$, for $n \in N$, so $F \cap Z_n = \emptyset$, where $Z_n = \bigcup\{\overline{X(B)}\colon B \in \mathcal{B}_n\} = \overline{Z_n}$. Then we have $p_S^{-1}(p_S(X) \setminus p_S(F)) \subset Z = \bigcup\{Z_n\colon n \in N\} \subset X \setminus F$. $\square$

Proof of Theorem 1. We prove the theorem by the transfinite induction with respect to $ht_{\sigma_p DC}(X)$. If $ht_{\sigma_p DC}(X)$ is a limit ordinal number then the conclusion of the theorem follows from the inductive assumption and Lemma 7. Let us assume that $ht_{\sigma_p DC}(X) = \beta + 1$. By Lemma 7, we can assume without loss of generality, that $X^{\beta}_{\sigma_p DC} = F = \bigcup\{F_{nt}\colon t \in T_n,\ n \in N\}$, the family $\{F_{nt}\colon\ t \in T_n\}$ is discrete and consists of compact sets and $\{F_n\colon\ n \in N\}$, for $F_n = \bigcup\{F_{nt}\colon\ t \in T_n\}$, is pairwise disjoint. We prove the theorem by showing that X is a union described in Lemma 6. If weight of X is equal to $|\alpha|$ then we can define an embedding $f : X \to (J(\alpha))^\alpha$ such that $\{p_\omega f(F_n)\colon\ n \in N\}$ is a closed family in $(J(\alpha))^\omega$ consisting of pairwise disjoint sets. In order to simplify notation we assume that X is already a subspace of $(J(\alpha))^\alpha$. Let us observe that $X = \bigcup\{p_\omega^{-1} p_\omega(F_n)\colon\ n \in N\} \cup \bigcup\{Z_n\colon\ n \in N\}$, where Z_n, for $n \in N$ satisfy the conclusion of Lemma 8 for $S = \omega$. By the inductive assumption the first player of the $G(DC, Z_n)$ game has a winning strategy for $n \in N$. Since $\{p_\omega(F_n)\colon\ n \in N\}$ is a closed pairwise disjoint family, we infer that $ht_{\sigma_p DC}(p_\omega^{-1} p_\omega(F_n)) = \beta + 1$ and $p_\omega^{-1} p_\omega(F_n))^{(\beta)}_{\sigma_p DC}$ is a discrete union of compact sets as a closed subset of F_n or $(p_\omega^{-1} p_\omega(F_n))^{(\beta)}_{\sigma DC} = \emptyset$. In both cases we conclude that the first player of the $G(DC, p_\omega^{-1} p_\omega(F_n))$ game has a winning strategy. By Lemma 6 we conclude that the theorem holds. $\square$

Proof of Theorem 2. We prove the theorem by the transfinite induction with respect to $ht_{\sigma DC}(X)$. If $ht_{\sigma DC}(X) = 0$ then $X = \emptyset$ and the conclusion of the theorem holds. Let us assume that the theorem holds for all spaces Z such that $ht_{\sigma DC}(Z) < ht_{\sigma DC}(X)$. If $ht_{\sigma DC}(X)$ is a limit ordinal number then the conclusion of the theorem follows from the inductive assumption and Lemma 6. Let us assume that $ht_{\sigma DC}(X) = \beta + 1$. By virtue of Lemma 2 we can assume, without loss of generality, that $(X)^{(\beta)}_{\sigma DC} = F = \bigcup\{F_n\colon\ n \in N\}$, where $F_n = \bigcup\{F_{ns}\colon\ s \in S_n\}$ and $\{F_{ns}\colon\ s \in S_n\}$ is a discrete family consisting of compact sets.

Let us note that we can restrict our attention to the case when $X^{(\beta)}_{\sigma DC}$ is a σ-compact set. Indeed, there is a pairwise disjoint partition $\{P_t\colon\ t \in T\}$ of $\bigcup\{S_n\colon\ n \in N\}$ consisting of countable or finite sets such that if $t \neq t'$, $s \in S_n \cap P_t$ and $s' \in S_{n'} \cap P_{t'}$, for certain n and n' in N, then $F_{ns} \cap F_{n's'} = \emptyset$. Now we can assume, changing eventually a little bit the embedding of X into $J(\tau)^{\omega_1}$, that the projection p_ω has the following properties:

(i) $p_\omega(F_{ns}) \cap p_\omega(F_{n's'}) = \emptyset$, provided that $s \in S_n \cap P_t$, $s' \in S_{n'} \cap P_{t'}$ and $t \neq t'$,
(ii) $\{p_\omega(F_{ns})\colon\ s \in S_n\}$ is a discrete family in $p_\omega(J(\tau)^{\omega_1})$ and $p_\omega(F_{ns}) \cap p_\omega(F_{ns'}) = \emptyset$, provided that $s \neq s'$, for $n \in N$.

By virtue of Lemmas 6 and 8, in order to prove the theorem it is enough to show that $p_\omega^{-1}(F_{ns})$, for $s \in S_n$ and $n \in N$ satisfy the conclusion of the theorem. To finish the proof of our statement we observe that $p_\omega^{-1} p_\omega(F_{ns})^{(\beta)}_{\sigma DC}$ is σ-compact, for $s \in S_n$ and $n \in N$.

In order to simplify our notation we assume that $X^{(\beta)}_{\sigma DC} = \bigcup\{F_n\colon\ n \in N\} = F$ and F_n is compact and, replacing eventually F_n by $\bigcup\{F_i\colon\ i = 1, 2, \dots, n\}$, that $F_n \subset F_{n+1}$, for $n \in N$.

Let M_1 be the following topological space. As a set M_1 is equal to X. The topology on M_1 is generated by $\mathcal{T} \cup \{\{x\}\colon x \in X \setminus F\} \cup \{p_\alpha^{-1} p_\alpha(F_n)\colon\ \alpha < \omega_1,\ n \in N\}$, where $\mathcal{T}$ is the topology on X.

Claim A. *The space M_1 is paracompact.*

Proof. Let $\mathcal{V}$ be an open cover of M_1. It is enough to show that F is a Lindelof subspace of M_1. From the fact that $F_n \subset F_{n+1}$, for $n \in N$, it follows that for every $x \in F_1$ there is an open H_x in X and $\alpha_x < \omega_1$ such that the family $\{H_x \cap p_{\alpha_x}^{-1} p_{\alpha_x}(F_1)\colon x \in F_1,\ H_x \in \mathcal{T}\}$ refines $\mathcal{V}$. Since F_1 is compact, we can find a countable subfamily $\mathcal{V}_1$ of $\mathcal{V}$ and $\alpha_1 < \omega_1$ such that $p_{\alpha_1}^{-1} p_{\alpha_1}(F_1) \subset \bigcup \mathcal{V}_1$. Let us assume that we have found $\alpha_1 < \alpha_2 < \dots < \alpha_n < \omega_1$ and a countable families $\mathcal{V}_1 \subset \mathcal{V}_2 \subset \dots \subset \mathcal{V}_n$ of $\mathcal{V}$ such that $\bigcup\{p_{\alpha_i}^{-1} p_{\alpha_i}(F_i)\colon\ i = 1, 2, \dots, j\} \subset \mathcal{V}_j$ for $1, 2, \dots, n$. Let us note that $I = F_{n+1} \setminus p_{\alpha_n}^{-1} p_{\alpha_n}(F_n)$ is a Lindelof subspace of M_1. Indeed, if $z \in I$ and $z_1 = p_{\alpha_n}(z)$ then there is an open $z_1 \in H$ in $p_{\alpha_n}(F_{n+1})$ which is an open subset of $p_{\alpha_n}(M_1)$ and a finite $\mathcal{V}(H)$ in $\mathcal{V}$ such that $p_{\alpha_n}^{-1}(H) \cap F_{n+1} \subset \bigcup \mathcal{V}(H)$. Since $p_{\alpha_n}(F)$ satisfies the second axiom of countability, where $p_{\alpha_n}(F)$ is considered with the topology generated by $\{p_{\alpha_n}(F_i)\colon\ i \in N\}$ and open sets of $p_{\alpha_n}(F)$, we conclude that there exists a countable subfamily $\mathcal{V}_{n+1}$ of $\mathcal{V}$ and $\alpha_n < \alpha_{n+1} < \omega_1$ such that $\mathcal{V}_n \subset \mathcal{V}_{n+1}$ and $p_{\alpha_{n+1}}^{-1} p_{\alpha_{n+1}}(F_{n+1}) \subset \bigcup \mathcal{V}_{n+1}$. If $\alpha_0 = \sup\{\alpha_n\colon\ n \in N\}$ then $\bigcup\{F_n\colon\ n \in N\} \subset p_{\alpha_0}^{-1} p_{\alpha_0}(\bigcup\{F_n\colon\ n \in N\}) \subset \bigcup \bigcup\{\mathcal{V}_n\colon\ n \in N\}$ and this completes the proof of Claim A. $\square$

Let E_α, for $\alpha \in Lim$, be a subset of $(J(\tau))^\alpha$ such that if $z \in E_\alpha$ then $z|\gamma \in p_\gamma(F)$, for $\gamma < \alpha$, and $z \notin p_\alpha(F)$. Put $E = \bigcup\{E_\alpha\colon\ \alpha \in Lim\}$ and $M_2 = X \cup E$. The base of the topology on M_2 is

$$\begin{aligned}\mathcal{B} = \bigl\{\{x\}\colon\ x \in X\bigr\} \cup \bigl\{B\bigl(y, (\gamma_1, \gamma_2, \dots, \gamma_n), U, (F_{i_1}, F_{i_2}, \dots, F_{i_n})\bigr)\colon\ U \in \mathcal{B}_{\gamma_n},\ y \in E_\alpha,\\ \gamma_1 < \gamma_2 < \dots < \gamma_n < \alpha,\ i_1 < i_2 < \dots < i_n,\ (i_j)_{j=1}^n \in N^n,\ n \in N\bigr\},\end{aligned}$$

where $\mathcal{B}_{\gamma_n}$ is a standard base in $p_{\gamma_n}(X)$ and $B(y, (\gamma_i)_{i=1}^n, U, (F_{i_j})_{j=1}^n) = \{y'\colon\ \bigcup\{E_{\gamma'}\colon\ \gamma' \in (\gamma_n, \omega_1) \cap Lim\}\colon\ y'|\gamma_j \in p_{\gamma_j}(F_{i_j}) \cap p_{\gamma_j}(U),\ j = 1, 2, \dots, n\} \cup \{x \in X \setminus p_\alpha^{-1}(y)\colon\ p_{\gamma_j}(x) \in p_{\gamma_j}(F_{i_j}) \cap p_{\gamma_j}(U),\ j = 1, 2, \dots, n\}$.

Claim B. *The space M_2 is paracompact.*

Proof. Put $H(B(y, (\gamma_i)_{i=1}^n, U, (F_{i_k})_{k=1}^n)) = B(y, (\gamma_i)_{i=1}^n, U, (F_{i_j})_{j=1}^n) \cap X$. Let $\mathcal{U}$ be an open cover of M_2. We can assume that $\mathcal{U} \subset \mathcal{B}$.

We say that a subspace R of M_1 is W-scattered if for every closed subset D of R there is a point $x \in D$ and a closed neighborhood V of x in D such that $V \cap F \subset F_n$, for a certain $n \in N$. In order to continue the proof we need:

Claim C. *The space $R = M_1 \setminus \{U \cap X\colon U \in \mathcal{U},\ U \cap E \neq \emptyset\}$ is W-scattered.*

Proof. Let us suppose that R is not W-scattered. Then there is a closed subset $R_1 \subset R$ such that for every $x \in R_1$ and every neighborhood V of x in R and $n \in N$ the intersection $(V \setminus F_n) \cap F \neq \emptyset$. Since $(F_n)_{n=1}^{\infty}$ is an increasing sequence of compact sets, we can define a sequence $(x_n)_{n=1}^{\infty} \subset R_1$ and $y \in E_\alpha$, for a certain $\alpha \in \mathit{Lim}$ such that $(x_n)_{n=1}^{\infty}$ converges to y in M_2. This contradicts the definition of R because for certain $U \in \mathcal{U}$ we have $y \in U$ and consequently $\{x_n\colon n \in N\} \cap U \neq \emptyset$. This completes the proof of Claim C. □

We shall finish the proof of Claim B by the transfinite induction with respect to $ht_W(R)$. If $ht_W(R) = 0$ then $\bigcup\{U \cap X\colon U \in \mathcal{U},\ U \cap E \neq \emptyset\} = M_1$. Then (see the proof of Claim A) there are $V_k \in \mathcal{U}$, for $k \in N$ and $\beta_0 < \omega_1$ such that each V_k is of the form $B(y_k, (\gamma_i)_{i=1}^{n_k}, U_k, (F_{i_j})_{j=1}^{k})$, $\sup\{\gamma_{n_k}\colon k \in N\} < \beta_0$, for a certain $\beta_0 < \omega_1$ and $p_{\beta_0}^{-1} p_{\beta_0}(\bigcup\{F_j\colon j \in N\}) \subset \bigcup\{V_k \cap X\colon k \in N\}$. Let us note that $\bigcup\{E_\alpha\colon \alpha \in (\beta_0, \omega_1) \cap \mathit{Lim}\} \subset \bigcup\{V_k\colon k \in N\}$, $\bigcup\{E_\alpha\colon \alpha \in (\beta_0 + 1) \cap \mathit{Lim}\}$ is a subspace of M_2 satisfying the second axiom of countability and $M_2 \setminus E = X$ is a discrete subspace of M_2. Hence $\mathcal{U}$ has a σ-locally finite open refinement. Let us suppose that $ht_W(R) = \lambda > 0$.

Case 1. λ is a limit ordinal number.

There are open subsets G_k of M_1, for $k \in N$, of the form $G_k = \{x \in M_1\colon x \in \bigcap_{j=1}^{n_k} p_{\gamma_{i_j}}^{-1} p_{\gamma_{i_j}}(F_{i_j}) \cap U_k\}$, where $(i_j)_{j=1}^{n_k}$ and $(\gamma_{i_j})_{j=1}^{n_k}$ are increasing sequences, $U_k \in \mathcal{B}_{\gamma_{i_{n_k}}}$ and $\beta_0 > \sup\{\gamma_{i_{n_k}}\colon k \in N\}$, such that $ht_W(P_k \cap R) < \lambda$, for $P_k = \{x \in M_1\colon x \in \bigcap_{j=1}^{n_k} p_{\gamma_{i_j}}^{-1} p_{\gamma_{i_j}}(F_{i_j}) \cap \overline{U_k}\}$ and

$$\bigcup\left\{p_{\beta_0}^{-1} p_{\beta_0}(F_n)\colon n \in N\right\} \subset \bigcup\{G_k\colon k \in N\}. \tag{$*$}$$

Put $M_2(P_k) = \{y \in \bigcup\{E_\gamma\colon \gamma \in (\gamma_{i_{n_k}}, \omega_1)\}\colon y|\gamma_{i_{n_k}} \in p_{\gamma_{i_{n_k}}}(P_k)\} \cup \bigcup p_{\gamma_{i_{n_k}}}^{-1} p_{\gamma_{i_{n_k}}}(P_k)$, $M_2(G_k) = \{y \in \bigcup\{E_\gamma\colon \gamma \in (\gamma_{i_{n_k}}, \omega_1) \cap \mathit{Lim}\}\colon y|\gamma_{i_{n_k}} \in p_{\gamma_{i_{n_k}}} \cup p_{\gamma_{i-n_k}}^{-1} p_{\gamma_{i_{n_k}}}(G_k)$. Let us note that $M_2(P_k)$ and $M_2(G_k)$ are closed and open sets, respectively, in M_2. From $(*)$ it follows that

$$\bigcup\left\{E_\gamma\colon \gamma \in (\beta, \omega_1) \cap \mathit{Lim}\right\} \subset \bigcup\left\{M_2(G_k)\colon k \in N\right\}. \tag{$**$}$$

By the inductive assumption $\mathcal{U}|M_2(P_k)$, where $\mathcal{U}|M_2(P_k)$ stands for the restriction of $\mathcal{U}$ to $M_2(P_k)$, has a σ-locally finite open refinement $\mathcal{V}(M_2(P_k))$ of $\mathcal{U}$ covering $M_2(P_k)$. Then $(**)$ implies that $\mathcal{V}_1 = \bigcup\{\mathcal{V}(M_2(P_k))|M_2(G_k)\colon k \in N\}$ is a σ-locally finite family covering $\bigcup\{E_\gamma\colon \gamma \in (\beta, \omega_1) \cap \mathit{Lim}\}$. Since $E \setminus \bigcup \mathcal{V}_1$ is a space satisfying the second axiom of countability, we conclude that $\mathcal{U}$ has a σ-locally finite refinement covering M_2.

Case 2. $\lambda = \lambda_0 + 1$.

There are open subsets G_k and O_l in M_1, for $k, l \in N$ of the form $G_k = \bigcap_{j=1}^{n_k} p_{\gamma_{i_j}}^{-1}(F_{i_j}) \cap U_k$, and $O_l = \bigcap_{j=1}^{n_l} p_{\beta_{i_j}}^{-1} p_{\beta_{i_j}}(F_{i_j}) \cap U'_l$, where sequences $(i_j)_{j=1}^{n_k} \in N^k$, $(i_j)_{j=1}^{n_l} \in N^l$, $(\gamma_{i_j})_{j=1}^{n_k} \in \omega_1^{n_k}$, $(\beta_{i_j})_{j=1}^{n_l} \in \omega_1^{n_l}$ are increasing and $U_k \in \mathcal{B}_{\gamma_{i_k}}$, $U'_l \in \mathcal{B}_{\beta_{i_{n_l}}}$ such that there is $\beta > \sup\{\gamma_{i_{n_k}}, \beta_{i_{n_l}}\colon k, l \in N\}$, $(P_k \cap R)_W^{(\lambda_0)} \subset F_{m_k}$, for certain $m_k \in N$ and $P_k = \bigcap_{j=1}^{n_k} p_{\gamma_{i_j}}^{-1} p_{\gamma_{i_j}}(F_{i_j}) \cap \overline{U_k}$, $\bigcup_{n=1}^{\infty} p_\beta^{-1} p_\beta(F_n) \cap R_W^{(\lambda_0)} \subset \bigcup\{G_k;\ k \in N\}$, $ht_W(P'_l \cap R) < \lambda$, for $P'_l = \bigcap_{j=1}^{n_l} p_{\gamma_{i_j}}^{-1} p_{\gamma_{i_j}}(F_{i_j}) \cap \overline{U_l}$ and

$$\bigcup_{n=1}^{\infty} p_\beta^{-1} p_\beta(F_n) \subset \bigcup_{k=1}^{\infty} G_k \cup \bigcup_{l=1}^{\infty} O_l. \tag{$***$}$$

Applying reasoning used in the previous case we conclude that it is enough to show that $\mathcal{U}|M_2(P_i)$ and $\mathcal{U}|M_2(P'_i)$, for $i \in N$ have σ-locally finite refinements. The family $\mathcal{U}|M_2(P'_i)$ has a σ-locally finite refinement by the inductive assumption. Let us fix $k \in N$. Put $A_\alpha = \{y \in M_2\colon y \in \bigcup\{E_\gamma\colon \gamma \in (\alpha, \omega_1) \cap \mathit{Lim}\},\ y|\alpha \in p_\alpha(F_{m_k})\} \cup p_\alpha^{-1} p_\alpha(F_{m_k}) \cap X$, for $\alpha < \omega_1$. Then $(A_\alpha)_{\alpha < \omega_1}$ is a decreasing sequence of clopen subsets of M_2, $\bigcap_{\alpha < \omega_1} A_\alpha \subset M_2 \setminus E$, and, for each $\alpha \in \mathit{Lim}$, $A_\alpha = \bigcap_{\gamma < \alpha} A_\gamma$. Put $B_\alpha = A_\alpha \setminus A_{\alpha+1}$. Then $\{B_\alpha\colon \alpha < \omega_1\}$ is a discrete clopen family in M_2 and $E \subset \bigcup_{\alpha < \omega_1} B_\alpha$. Since $M_2(P_k) \cap X = P_k$, $(P_k \cap R)_W^{(\lambda_0)} \subset F_{m_k}$, $B_\alpha \cap F_{m_k} = \emptyset$, for $\alpha < \omega_1$, we infer that $(P_k \cap R \cap B_\alpha)_W^{(\lambda_0)} = \emptyset$, so by the inductive assumption $\mathcal{U}|M_2(P_k) \cap B_\alpha$ has a σ-locally finite open refinement, for $\alpha < \omega_1$ and consequently $\mathcal{U}|M_2(P_k)$ has a σ-locally finite refinement. This completes the proof of Claim B. □

Let $Y_1 = X_F$. From the assumption of the theorem and Lemma 4 it follows that $Y_1 \times M_2$ is paracompact. We shall show that the last fact implies that the first player has a winning strategy in the $G(DC, X)$ game. Let us consider two disjoint closed sets $K_1 = Y_1 \times E$ and $K_2 = \{(x,x) \in Y_1 \times M_2\colon\ x \in X\}$ in $Y_1 \times M_2$. Then there are open a disjoint sets U_1 and U_2 in $Y_1 \times M_2$ such that $K_1 \subset U_1$ and $K_2 \subset U_2$. For every $x \in Y_1$. Let U_x be an open set in Y_1 such that $U_x \times \{x\} \subset U_2$. We shall define a winning strategy for the first player in the $G(DC, X)$ game. If $R \subset X$ is a closed subset of X and the first player in the $G(DC, R)$ game has a winning strategy s_R then we denote the sets chosen by the players by $Z_n(s_R)$. Let $\mathcal{K}$ be a family of sets of the form $K = \bigcap_{i=1}^n p_{\gamma_i}^{-1} p_{\gamma_i}(F_{k_i}) \cap \overline{U_K}$, where sequences $(\gamma_i)_{i=1}^n \in \omega_1^n$ and $(k_i)_{i=1}^n \in N^n$ are increasing, $U_K \in \mathcal{B}_{\gamma_n}$, where $\mathcal{B}_{\gamma_n}$ is a standard base in $p_{\gamma_n}(X)$ and $\overline{U_K}$ is the closure of U_K in $p_{\gamma_n}(X)$. Put $\alpha(K) = \gamma_n$.

Put $M_{1_\alpha} = p_\alpha(F)$ with the topology generated by $\{p_\alpha(F_n)\colon\ n \in N\}$ and open sets of $p_\alpha(F)$. Then M_{1_α} is a regular space satisfying the second axiom of countability. Let d_α be a metric on M_{1_α} compatible with the topology of M_{1_α}. If $K \subset M_{1_\alpha}$ then $diam_{d_\alpha}(K)$ stands for the diameter of K with respect to d_α. There exists a countable subfamily $\mathcal{D}_1$ of $\mathcal{K}$ and $\alpha_1 < \omega_1$ such that $\mathcal{D}_1$ refines $\{U_x\colon\ x \in F\}$, $\alpha(K) < \alpha_1$, for $K \in \mathcal{D}_1$ and $\bigcup \mathcal{D}_1 \supset p_{\alpha_1}^{-1} p_{\alpha_1}(F)$ (see the proof of Claim A). For each $K \in \mathcal{D}_1$ fix x_K such that $K \subset U_{x_K}$. By virtue of Lemma 8 there is a family $\mathcal{J}_1 = \{J_{1n}\colon\ n \in N\}$, closed in X and such that $p_{\alpha_1}^{-1}(p_{\alpha_1}(X) \setminus p_{\alpha_1}(F)) \subset \bigcup \mathcal{J}_{\alpha_1} \subset X \setminus F$.

Put $Z_0 = X$ and the set chosen by the first player is defined by the formula $Z_1 = F_1 \cup Z_1(s_{J_{11}})$. Let Z_2 be the set chosen by the second player. Let us assume that $(Z_i)_{i=1}^{2n}$, $(\mathcal{D})_i^n$, an increasing sequence $(\alpha_i)_{i=1}^n \in \omega_1^n$ and $\mathcal{J}_i = \{J_{ik}\colon\ k \in N\}$ have been defined, for $i = 1, 2, \ldots, n$, in such a way that:

(1) Z_i for $i = 1, 2, \ldots, 2n$ are defined according to the rules of the $G(DC, X)$ game.

(2) $\mathcal{D}_i$ is a countable subfamily of $\mathcal{K}$, $\alpha(K) < \alpha_i$, for each $K \in \mathcal{D}_i$ and $p_{\alpha_i}^{-1} p_{\alpha_i}(F) \subset \bigcup \mathcal{D}_i$, for $i = 1, 2, \ldots, n$.

(3) J_{ik}, for $k \in N$, is a closed subset of X such that $p_{\alpha_i}^{-1}(p_{\alpha_i}(Z_{2(i-1)}) \setminus p_{\alpha_i}(F)) \subset \bigcup \mathcal{J}_i \subset X \setminus F$.

(4) If $1 < i \leqslant n$, $K \in \mathcal{D}_{i-1}$ then there is a countable subfamily $\mathcal{D}(K)$ of $\mathcal{K}$ and $\alpha_i > \sup\{\alpha_{i-1},\ \alpha(K')\colon\ K' \in \mathcal{D}(K)\}$, $p_{\alpha_{i-1}}(K') \subset p_{\alpha_{i-1}}(K)$, $diam_{d_{\alpha_j}} p_{\alpha_j}(K') < 1/i$, for $K' \in \mathcal{D}(K)$, $j = 1, 2, \ldots, i-1$, $\bigcup \mathcal{D}(K) \supset p_{\alpha_i}^{-1} p_{\alpha_i}(F \cap K \cap Z_{2(i-1)})$ and $\mathcal{D}(K)$ refines $\{U_x\colon\ x \in F \cap K \cap Z_{2(i-1)}\}$.

(5) If $K' \in \mathcal{D}(K)$ then it is fixed $x_{K'} \in F \cap K \cap M_{2(i-1)}$ such that $K' \subset U_{x_{K'}}$.

(6) If $1 < i \leqslant n$, $K \in \mathcal{D}_{i-1}$ then there is a closed family $\mathcal{J}(K) = \{J_k(K)\colon\ k \in N\}$ in K such that $p_{\alpha_i}^{-1}(p_{\alpha_i}(K \cap Z_{2(i-1)}) \setminus p_{\alpha_i}(K \cap Z_{2(i-1)} \cap F)) \subset \bigcup \mathcal{J}(K) \subset X \setminus F$.

(7) If $1 < i \leqslant n$ then $\mathcal{J}_i = \{\mathcal{J}(K)\colon\ K \in \mathcal{D}_{i-1}\}$ and $\mathcal{D}_i = \bigcup \{\mathcal{D}(K)\colon\ K \in \mathcal{D}_{i-1}\}$.

(8) $\bigcup \mathcal{D}_i \cup \bigcup_{j=1}^i \bigcup \mathcal{J}_j \supset Z_{2(i-1)}$.

(9) If $1 \leqslant i \leqslant n$ then

$$Z_{2i-1} = \left(\bigcup_{j=1}^{i} F_i\right) \cap Z_{2(i-1)} \cup \left(\bigcup_{k=1}^{i} \bigcup_{j=1}^{i-k+1} Z_{2(i-j-k+2)-1}(s_{J_{kj}})\right) \cap Z_{2(i-1)}.$$

Conditions (1)–(9) describe how to define $Z_{2(n+1)}$, $\mathcal{D}_{n+1}$ and $\mathcal{J}_{n+1}$.

Let us assume that there exists $x_0 \in \bigcap \{Z_{2n}\colon\ n \in N\}$. From $\bigcap \{Z_{2n}\colon\ n \in N\} \cap F = \emptyset$ it follows that $x_0 \in X \setminus F$. Let us note that if $J = J_{ik}$, for certain $i, k \in N$, then J is a closed set missing F. From the definition of Z_n, it is easy to see that for odd indices starting with a certain $n_0 \in N$ the sequence $(Z_{2(n_0-1)+k})_{k=1}^\infty$ realizes the winning strategy in the $G(DC, J)$ game for the first player. Hence we conclude that $x_0 \notin \bigcup_{n=1}^\infty \bigcup \mathcal{J}_n$. From this fact it follows that we can define sequences $(x_n)_{n=1}^\infty \in X^\infty$, $(K_n)_{n=1}^\infty \in (\mathcal{D}_n)_{n=1}^\infty$ such that:

(a) $x_0 \in \bigcap_{n=1}^\infty K_n$,
(b) $K_{i+1} \in \mathcal{D}(K_i)$, for $i \in N$,
(c) $x_n = x_{K_n}$, for $n \in N$, meaning that $K_n \subset U_{x_n}$,
(d) $x_{n+1} \in K_n$; (in fact, (d) follows from (b)).

From (4) we infer that if $\alpha = \sup\{\alpha_n\colon\ n \in N\}$ then $X \setminus F \supset \bigcap_{n=1}^\infty K_n = p_\alpha^{-1}(\{x_0|\alpha\})$. Let us note that $p_{\beta_0}^{-1} p_{\beta_0}(x_0) \cap F \neq \emptyset$, for every $\beta_0 < \alpha$. If $p_{\beta_0}^{-1} p_{\beta_0}(x_0) \cap F = \emptyset$, for a certain $\beta_0 < \alpha$, then there is $n \in N$ such that $p_{\alpha_n}^{-1} p_{\alpha_n}(x_0) \subset X \setminus F$ and by virtue of (3) $x_0 \in \bigcup \mathcal{J}_n$, contradiction with one of our previous observation. Hence we conclude that $y = x_0|\alpha \in E_\alpha$. From (a)–(d) and (4) it follows that the sequence $((x_0, x_n))_{n=1}^\infty$ converges in $Y_1 \times M_2$ to $(x_0, y) \in K_1$. The last fact contradicts the assumption that U_1 and U_2 are open disjoint sets containing, respectively, K_1 and K_2. □

Proof of Theorem 3. We prove the theorem by transfinite induction with respect to $ht_G(X)$. If $ht_G(X)$ is 0 or a limit ordinal number then the conclusion of the theorem follows from Lemma 7 and the inductive assumption. Let us assume that $ht_G(X) = \beta + 1$. By virtue of Lemma 7 we can assume, without loss of generality, that $X_G^{(\beta)} = F$ and the first player in the $C(DC, F)$ game has a winning strategy. Since the projection $p_\alpha : X \to J(\tau)^\alpha$ is closed and $p_\alpha(X) = X_\alpha$ satisfies the first axiom of countability, by Lemmas 1 and 2 there exists a closed subset F_α of X such that $f_\alpha = p_\alpha|F_\alpha$ is perfect mapping, $p_\alpha(X) = X_\alpha = p_\alpha(F_\alpha)$ and $X_\alpha = \bigcup \{Z_{ns}\colon\ s \in S_{n\alpha},\ n \in N\}$, where $\{Z_{ns}\colon\ s \in S_{n\alpha}\}$ is a discrete family of compact sets, for $n \in N$, see Lemma 2. Put $F_{ns} = f_\alpha^{-1}(Z_{ns})$ for $s \in S_{n\alpha}$ and $n \in N$. Then $\{F_{ns}\colon\ s \in S_{n\alpha}\}$ is a discrete family of compact sets in X, for $n \in N$.

If α is a limit ordinal number then C_α is a subset of X_α satisfying the following condition:

(∗) if $z \in C_\alpha$ then $p_\gamma^{-1}(z|\gamma) \cap F \neq \emptyset$, for every $\gamma < \alpha$, and $p_\alpha^{-1}(z) \subset X \setminus F$.

Put $C = \bigcup\{C_\alpha\colon \alpha \in Lim\}$ and $Y = X \cup C$. The topology on Y is the following one: points of X are isolated and the base at $z \in C_\alpha$ consists of the sets of the form

$$U(z,\lambda) = \left\{z' \in \bigcup\{C_\gamma\colon \lambda < \gamma\}\colon z'|\lambda = z|\lambda\right\} \cup \left\{x \in X \setminus p_\alpha^{-1}(z)\colon x|\lambda = z|\lambda\right\},$$

for $\lambda < \alpha$. Then Y is paracompact. Indeed, if $\mathcal{U}$ is an open cover of Y then for every $z \in C$ there is the least ordinal number $\lambda(z)$ for which we can find $U \in \mathcal{U}$ such that $U(z,\lambda(z)) \subset U$. Let us note that if $z\ z' \in C$ and $z \neq z'$ then the sets $U(z,\lambda(z))$ and $U(z',\lambda(z'))$ are equal or disjoint. Hence we can find a pairwise disjoint open family $\mathcal{V}$ refining $\mathcal{U}$ and such that $C \subset \bigcup \mathcal{V}$. Since $Y \setminus \bigcup \mathcal{V}$ is discrete, we conclude that Y is paracompact. By Lemma 4, the product $X_F \times Y$ is normal and the sets $A_1 = X_F \times C$ and $A_2 = \{(x_1,x_2) \in X^2\colon x_1 = x_2\}$ are disjoint and closed sets in $X_F \times Y$. There are open and disjoint sets G_1 and G_2 such that $A_1 \subset G_1$ and $A_2 \subset G_2$.

We shall show that the last fact implies that the first player of the $G(DC,X)$ game has a winning strategy.

For each $x \in X_F$ fix an open set U_x in X_F such that $U_x \times \{x\} \subset G_2$. We can assume that U_x, for $x \in F$, belongs to a standard base. Hence there are a finite set $T_x \subset \omega_1$ and open sets $U_x(t) \subset J(\tau)$, for $t \in T_x$ such that $U_x = \{x' \in X\colon x'(t) \in U_x(t),\ t \in T_x\}$.

We shall define a winning strategy for the first player in the $G(DC,X)$ game. The sets chosen by the players will be denoted by M_n, for $n \in N$. Put $M_0 = X$. Let $\mathcal{D}_1$ be a closed locally finite family in X such that $\mathcal{D}_1$ refines $\{U_x\colon x \in F\}$, $D \cap F \neq \emptyset$, for $D \in \mathcal{D}_1$, and $\bigcup\{Int_X D\colon D \in \mathcal{D}_1\} \supset F$, where $Int_X(D)$ stands for the interior of D with respect to X. For each $D \in \mathcal{D}_1$ fix $x_D \in F$ and $\alpha_D \in \omega_1$ such that $D \subset U_{x_D}$ and $T_{x_D} \subset \alpha_D$.

If R is a closed subset of X and the first player in the $G(DC,R)$ has a winning strategy then we denote by $M_n(s_R)$ the sets chosen by the players according to this strategy.

Put $E = X \setminus \bigcup\{Int_X D\colon D \in \mathcal{D}_1\}$, $\mathcal{E}_1 = \{E\}$ and $\mathcal{E}_0 = \emptyset$. Put $M_1 = M_1(s_F) \cup M_1(s_E) \cup \bigcup\{F_{1s} \cap D\colon s \in S_{1\alpha_D},\ D \in \mathcal{D}_1\}$ and let M_2 be a closed subset chosen by the second player. Let us assume that $(M_i)_{i=1}^{2n}$, $(\mathcal{D}_i)_{i=1}^{n}$ and $(\mathcal{E}_i)_{i=0}^{n}$ are defined in such a way that:

(1) $(M_i)_{i=1}^{2n}$ is defined according to the rules of the $G(DC,X)$ game and $M_0 = X$.

(2) If $1 \leqslant i \leqslant n$ then $\mathcal{D}_i$ and $\mathcal{E}_i$ are closed locally finite families and $\bigcup \mathcal{D}_i \cup \bigcup \mathcal{E}_i = M_{2(i-1)}$.

(3) $D \cap F \cap M_{2(i-1)} \neq \emptyset$ for each $D \in \mathcal{D}_i$ and $E \cap F \cap M_{2(i-1)} = \emptyset$ for each $E \in \mathcal{E}_i$.

(4) For every $D \in \mathcal{D}_i$ there is $x_D \in M_{2(i-1)} \cap F$ and $\alpha_D \in \omega_1$ such that $D \subset U_{x_D}$ and $T_{x_D} \subset \alpha_D$.

(5) If $2 \leqslant i \leqslant n$ then for every $D_0 \in \mathcal{D}_{i-1}$ there is a locally finite closed family $\mathcal{D}_{i-1}(D_0)$ of subsets of D_0 refining $\{U_x \cap D_0 \cap M_{2(i-1)}\colon x \in D_0 \cap M_{2(i-1)} \cap F\}$ and such that $\bigcup\{Int_{D_0 \cap M_{2(i-1)}} D\colon D \in \mathcal{D}_{i-1}(D_0)\} \supset D_0 \cap M_{2(i-1)} \cap F$, $D \cap M_{2(i-1)} \cap F \neq \emptyset$, for $D \in \mathcal{D}_{i-1}(D_0)$ and $E(D_0) = D_0 \setminus \bigcup\{Int_{D_0 \cap M_{2(i-1)}}(D)\colon D \in \mathcal{D}_{i-1}(D_0)\}$.

(6) If $2 \leqslant i \leqslant n$, $D_0 \in \mathcal{D}_{i-1}$ and $D \in \mathcal{D}_{i-1}(D_0)$ then $\alpha_{D_0} < \alpha_D$.

(7) if $2 \leqslant i \leqslant n$ then $M_{2i-1} = (M_{2i-1}(s_F) \cup \bigcup\{M_{2(i-j+1)-1}(s_E)\colon E \in \mathcal{E}_j \setminus \mathcal{E}_{j-1},\ j = 1,2,\dots,i\} \cup \bigcup\{F_{js} \cap D\colon s \in S_{j\alpha_D},\ j = 1, 2,\dots,i\}) \cap M_{2(i-1)}$.

(8) if $1 < i \leqslant n$ then $(\bigcup \mathcal{D}_i) \cap (\bigcup\{\bigcup \mathcal{E}_j\colon j = 0,1,2,\dots,i-1\}) = \emptyset$.

(9) If $1 < i \leqslant n$, $D_0 \in \bigcup\{\mathcal{D}_k\colon k = 1,2,\dots,i-1\}$, $s \in S_{j\alpha_{D_0}}$ and $j = 1,2,\dots,i-1$ then there is a finite set $O(Z_{js}(D_)) \subset p_{\alpha_{D_0}}(D_0) \cap Z_{js}$ such that $p_{\alpha_{D_0}}^{-1}(Z_{js} \cap p_{\alpha_{D_0}}(D_0)) \cap D_0 \cap M_{2(i-1)} \subset p_{\alpha_{D_0}}^{-1}(O(Z_{js}(D_0))) \cap D_0 \cap M_{2(i-1)}$.

(10) For each $o \in O(Z_{js}(D_0))$, $s \in S_{j\alpha_{D_0}}$, $j = 1,2,\dots,i-1$ and $D_0 \in \bigcup\{\mathcal{D}_j\colon j = 1,2,\dots,i-1\}$, if $R_1(o) = p_{\alpha_{D_0}}^{-1}(o) \cap D_0 \cap M_{2(i-1)} \cap F \neq \emptyset$ then there is a locally finite closed family $\mathcal{D}(o)$ in $R_2(o) = p_{\alpha_{D_0}}^{-1}(o) \cap D_0 \cap M_{2(i-1)}$ which refines $\{U_x \cap R_2(o)\colon x \in R_1(o)\}$ such that $R_1(o) \subset \bigcup\{Int_{R_2(o)} D\colon D \in \mathcal{D}(o)\} = Q(o)$ and $D \cap F \neq \emptyset$ for each $D \in \mathcal{D}(o)$. Put $E(o) = R_2(o) \setminus Q(o)$.

(11) If $D \in \mathcal{D}(o)$ then $x_D \in R_1(o)$ are fixed in such a way that $D \subset U_{x_D}$, $\alpha_{D_0} < \alpha_D$ and $T_{x_D} \subset \alpha_D$.

(12) If $R_1(o) = \emptyset$ then $\mathcal{D}(o) = \emptyset$ and $E(o) = R_2(o)$.

(13) If $1 < i \leqslant n$ then $\mathcal{D}_i = \bigcup\{\mathcal{D}_{i-1}(D_0)\colon D_0 \in \mathcal{D}_{i-1}\} \cup \bigcup\{\mathcal{D}(o)\colon o \in O(Z(_{js}(D_0))),\ s \in S_{j\alpha_{D_0}},\ j = 1,2,\dots,i-1,\ D_0 \in \bigcup_{j=1}^{j=i-1} \mathcal{D}_j\}$ and $\mathcal{E}_i = \mathcal{E}_{i-1} \cup \{E(D_0)\colon D_0 \in \mathcal{D}_{i-1}\} \cup \{E(o)\colon o \in O(Z_{js}(D_0)),\ s \in S_{j\alpha_{D_0}},\ j = 1,2,\dots,\ i-1,\ D_0 \in \bigcup_{j=1}^{j=i-1} \mathcal{D}_j\}$, for $i \geqslant 2$.

Conditions (1)–(13) describe how to define M_{2n+1}, $\mathcal{D}_{n+1}$ and $\mathcal{E}_{n+1}$. The set $M_{2(n+1)}$ is selected by the second player according to the rules of the $G(DC,X)$ game.

The existence of the set $O(Z_{js}(D_0))$ described in (9) follows from the facts that the projection $p_{\alpha_{D_0}}$ is closed, compactness of $F_{js} \cap D_0$, $p_{\alpha_{D_0}}(F_{js} \cap D_0) \subset Z_{js} \cap p_{\alpha_{D_0}}$ and $M_{2(i-1)} \cap M_{2(i-1)-1} = \emptyset$.

Let us assume that there exists $x_0 \in \bigcap_{n=1}^{\infty} M_{2n}$. Note that $F \cap M_{2n} = M_{2n}(s_F)$. Since the family $\bigcup_{n=1}^{\infty} \mathcal{E}_n$ consists of closed sets in X missing F and for every $E \in \mathcal{E}_n$, for $n \in N$, the sequence $(M_i \cap E)_{i=2n}^{\infty}$ realizes the winning strategy for the first player in the $G(DC,E)$ game, we conclude that $x_0 \notin \bigcup\bigcup\{E \in \mathcal{E}_n\colon n \in N\}$. Let $D_1 \in \mathcal{D}_1$ be such that $x_0 \in D_1$ and i_1 be the least natural number that $x_0|\alpha_{D_1} \in p_{\alpha_{D_1}} \cap Z_{i_1 s}$, for $s \in S_{i_1\alpha_{D_1}}$. By virtue of (10) and (11), for $o_1 = x_0|\alpha_{D_1}$, there is $D_2 \in \mathcal{D}(o_1) \subset \mathcal{D}_{i_1+1}$, $y_2 = x_{D_2}$ such that $y_2 \in p_{\alpha_{D_1}}^{-1}(0_1) \cap D_1 \cap M_{2i_1} \cap F$ and $D_2 \subset p_{\alpha_{D_1}}^{-1}(o_1) \cap D_1 \cap M_{2i_1} \cap U_{y_2}$. Continuing this process we define increasing sequences $(i_k)_{k=1}^{\infty} \in N^\omega$, $(\alpha_{D_k})_{k=1}^{\infty} \in \omega_1^\omega$, where $D_k \in \mathcal{D}_{i_k}$, $(y_k)_{k=1}^{\infty} \in F^\omega$ such that $U_{y_{k+1}} \supset p^{-1}_{\alpha_{D_k}} p_{\alpha_{D_k}}(x_0) \supset D_{k+1}$ and $y_{k+1} \in p^{-1}_{\alpha_{D_k}} p_{\alpha_{D_k}}(x_0) \cap D_k$. Put $\lambda_0 = \sup\{\alpha_{D_k}\colon k \in N\}$ and $z = x_0|\lambda_0$. Then for every $\gamma < \lambda_0$ we have $p^{-1}_{\gamma}(z|\gamma) \cap F \neq \emptyset$ and $p^{-1}_{\lambda_0}(z) \subset \bigcap_{n=2}^{\infty} U_{y_n}$.

In order to insure that $p^{-1}_{\lambda_0}(z) \cap F = \emptyset$ we have to modify a little bit the construction of $(M_i(s_F))_{i=1}^{\infty}$. The set $M_1(s_F)$ is unchanged. If $M_2(s_F)$ is chosen by the second player then we choose an open subset H_2 of X such that $M_2(s_F) \subset H_2 \subset \overline{H_2} \subset X \setminus M_1(s_F)$ and $M'_3(s_F) = s_F(M_1(s_F), \overline{H_2} \cap F)$ and $M_3(s_F) = M'_3(s_F) \cap M_2(s_F)$. If H_{2n} is defined then $H_{2(n+1)}$ is an open subset of X such that $M_{2(n+1)}(s_F) \subset H_{2(n+1)} \subset \overline{H_{2(n+1)}} \subset (X \setminus \bigcup\{M'_{2i+1}(s_F)\colon i = 1, 2, \ldots, n\}) \cap H_{2n}$ and $M'_{2(n+1)+1}(s_F) = s_F(M_1(s_F), \overline{H_2} \cap F, M'_3(s_F), \ldots, \overline{H_{2n}} \cap F, M'_{2n+1}(s_F), \overline{H_{2(n+1)}} \cap F)$ and $M_{2(n+1)+1}(s_F) = M'_{2(n+1)+1}(s_F) \cap M_{2(n+1)}(s_F)$. Then $\emptyset = F \cap \bigcap\{\overline{H_{2(n+1)}}\colon n \in N\} \supset \bigcap\{M_{2(n+1)}(s_F)\colon n \in N\}$. Now, if we assume that the sets U_x, appearing in the $i-th$ step of construction are, additionally, subsets of H_{2i} then we insure that $p^{-1}_{\lambda_0}(z) \cap F = \emptyset$ and consequently $z \in C_{\lambda_0}$. Let us observe that the sequence $((x_0, y_i))_{i=1}^{\infty} \subset G_2$ converges to $(x_0, z) \in A_1 \subset G_1$ in $X_F \times Y$ and this contradicts the fact that G_1 and G_2 are disjoint open sets. □

Question 1. Does Conjecture 1 hold for spaces whose set of accumulation points is σ-compact?

Remark 2. If X is a paracompact and separable subspace of I^{ω_1} such that for each $\alpha < \omega_1$ the projection $p_\alpha : X \to X_\alpha \subset I^\alpha$ is closed then $X \in \mathcal{P}$ if and only if X is σ-compact.

Proof. By Lemma 2 we infer that $X \in \mathcal{P}$ implies that for each $\alpha < \omega_1$ $p_\alpha(X) = X_\alpha \subset N^\alpha$ is σ-compact. Let $A = \{a_n\colon n \in N\}$ be a dense subset of X. There is $\alpha < \omega_1$ such that $p_\alpha|A$ is a one-to-one mapping. By Lemma 1 there is a closed set F such that $A \subset F$, meaning that $F = X$, $f_1 = p_\alpha|A$ is a perfect mapping and $f_1(X) = X_\alpha$. Since X_α σ-compact and f_1 is a perfect mapping we conclude that X is σ-compact. The other implication is trivial. □

Remark 3. In [1] we gave a characterization of ω_1-metrizable spaces which belong to $\mathcal{P}$. Let us observe that such spaces have, in a trivial way, closed projections onto initial countably many coordinates.

Remark 4. One can show that if $X \in \mathcal{P}$, $f : X \to X_1$ is a closed mapping onto a P-space X_1 of weight not greater than ω_1, then $X_1 \in \mathcal{P}$. In particular we have that if $X \subset N^{\omega_2}$ is a Lindelof P space and $X \in \mathcal{P}$ then for every $\alpha < \omega_2$ the projection $p_\alpha : X \to N^\alpha$ is closed, so $p_\alpha(X) \in \mathcal{P}$.

Remark 5. If X is a space then let us denote by X_ω the same set with the topology generated by G_δ subsets of X. Let us note that neither $(X \in \mathcal{P}) \Rightarrow (X_\omega \in \mathcal{P})$ nor $(X_\omega \in \mathcal{P}) \Rightarrow (X \in \mathcal{P})$ hold.

Proof. If $X = \{0,1\}^{\omega_1}$ then $X \in \mathcal{P}$ as a compact space and X_ω does not belong to $\mathcal{P}$ by the results of [1]. □

If we put $X = R_Q$ then X does not belong to $\mathcal{P}$ and $X_\omega \in \mathcal{P}$ as a discrete space.

Question 2. Let us assume that X is a P-space, of weight not greater than ω_2 and $X \in \mathcal{P}$. Is it true that $X_{\omega_1} \in \mathcal{P}$, where X_{ω_1} stands for the same set X with the topology generated by sets of the form $\bigcap\{G_\alpha\colon \alpha < \omega_1, G_\alpha \text{ is open in } X\}$.

Question 3. What is the characterization of spaces of weight not greater than ω_1 belonging to $\mathcal{P}$?

Question 4. Is the assumption that X is a G-scattered space essential in Theorem 3?

Question 5. Is the assumption that p_α is closed, for each $\alpha < \omega_1$, essential in Theorem 3?

Remark 6. If X satisfies all assumptions of Theorem 3 except the one that X is a G-scattered space then there exists an increasing sequence $(K_\alpha)_{\alpha<\omega_1}$ of closed subsets of X such that $f_\alpha = p_\alpha|K_\alpha$ is a perfect mapping for $\alpha < \omega_1$.

If the set

$$S = \left\{\alpha < \omega_1\colon K_\alpha \setminus \bigcup\{K_\beta\colon \beta < \alpha\} \neq \emptyset\right\}$$

is not stationary then the first player in the $G(DC, X)$ game has a winning strategy.

Proof. By Lemma 1 there exists an increasing sequence $(K_\alpha)_{\alpha<\omega_1}$ of closed sets in X such that $f_\alpha = p_\alpha|K_\alpha$ is a perfect mapping and $f_\alpha(K_\alpha) = p_\alpha(X) = X_\alpha$. The space X_α, as a perfect and metrizable image of an element of $\mathcal{P}$, is of the form $X_\alpha = \bigcup\{Z_{ns}\colon s \in S_{n\alpha},\ n \in N\}$, where $\{Z_{ns}\colon s \in S_{n\alpha}\}$ is a discrete family of compact sets. Then $F_{ns} = f^{-1}_\alpha(Z_{ns})$ is compact, for $s \in S_{n\alpha}$, as a perfect preimage of a compact set. Note that $K_\alpha = \bigcup\{F_{ns};\ s \in S_{n\alpha},\ n \in N\}$. Let us choose an unbounded and closed subset C of ω_1 missing S and take $\alpha_0 \in C$. Put

$$A_1 = \bigcup\{F_{1s}\colon\ s \in S_{1\alpha_0}\}$$

and let A_2 be a closed subset of X chosen by the second player. For each $s \in S_{1\alpha_0}$ put $D_s = p_{\alpha_0}^{-1}(Z_{1s}) \cap A_2$. Then there is a finite set $O(Z_{1s}) \subset Z_{1s}$ such that $p_{\alpha_0}^{-1}(Z_{1s}) \cap A_2 \subset p_{\alpha_0}^{-1}(O(Z_{1s})) \cap A_2$ (see the proof of Theorem 3). By the definition of A_2 we have $p_{\alpha_0}^{-1}(O(Z_{1s}) \cap A_2 \cap K_{\alpha_0} = \emptyset$. For $o \in O(Z_{1s})$ put $R(o) = p_{\alpha_0}^{-1}(o) \cap A_2$. If $R(o) \neq \emptyset$ then consider an open family $\mathcal{U}(o)$ in X covering $R(o)$ such that $(\bigcup\mathcal{U}(o)) \cap K_{\alpha_0} = \emptyset$ and for each $U \in \mathcal{U}(o)$ there is a finite set $T(U) \subset \omega_1$ such that $U = p_{T(U)}^{-1} p_{T(U)}(U)$. Let us observe that $U \cap p_{T(U)}^{-1} p_{T(U)}(K_{\alpha_0}) = \emptyset$ and consequently, for each $\alpha_0 < \alpha < \omega_1$, if $T(U) \subset \alpha$ then $U \cap p_\alpha^{-1} p_\alpha(K_0) = \emptyset$. Let us consider a closed locally finite cover $\mathcal{D}(o)$ of $R(o)$ which refines $\mathcal{U}(o)$. For each $D \in \mathcal{D}(o)$ fix $U(D) \in \mathcal{U}(o)$ and $\alpha_0 < \alpha(D) < \omega_1$ such that $T(U(D)) \subset \alpha(D) \in C$ and $D \subset U(D)$. Then we have $D \cap p_{\alpha(D)}^{-1} p_{\alpha(D)}(K_{\alpha_0}) = \emptyset$. Put $\mathcal{D}_1 = \bigcup\{\mathcal{D}(o)\colon\ o \in O(Z_{1s}),\ s \in S_{1\alpha_0}\}$. Then $\mathcal{D}_1$ is a closed locally finite family in X. Let us assume that we have defined a sequence $(A_i)_{i=1}^{2n}$ of closed sets in X satisfying the rules of the $G(DC, X)$ game, a locally finite closed family $\mathcal{D}_i$ in X and $\alpha(D) \in \omega_1 \cap C$, for $D \in \mathcal{D}_i$, $i = 1, 2, \dots, n$ in such a way that the following conditions hold:

(a) If $1 < i < n$ then $A_{2i-1} = f_{\alpha_0}^{-1}(\bigcup\{Z_{is}\colon\ s \in S_{i\alpha_0}\}) \cap A_{2i-2} \cup (\bigcup\{D \cap f_{\alpha(D)}^{-1}(Z_{js})\colon\ s \in S_{j\alpha(D)},\ j = 1, 2, \dots, i,\ D \in \mathcal{D}_{i-1}\}) \cap A_{2i-2}$.
(b) For each $1 < i \leqslant n$, $D \in \mathcal{D}_{i-1}$, $s \in S_{j\alpha(D)}$, $j = 1, 2, \dots, i$ there is a finite set $O(Z_{js}, D) \subset Z_{js} \cap p_{\alpha(D)}(D)$ such that $p_{\alpha(D)}^{-1}(Z_{js}) \cap A_{2i} \cap D \subset p_{\alpha(D)}^{-1}(O(Z_{js}, D)) \cap A_{2i} \cap D$.
(c) For each $o \in O(Z_{js}, D)$, $s \in S_{j\alpha(D)}$, $j = 1, 2, \dots, i$, and $R(o) = p_{\alpha(D)}^{-1}(o) \cap A_{2i} \cap D$ there is a locally finite closed cover of $R(o)$ and $\alpha(D) < \alpha(D') \in C$, for $D' \in \mathcal{D}(o)$ such that $A_{2i} \cap D' \cap p_{\alpha(D')}^{-1} p_{\alpha(D')}(K_{\alpha(D)}) = \emptyset$.
(d) For each $1 < i \leqslant n$ we have $\mathcal{D}_i = \bigcup\{\mathcal{D}_j\colon\ j = 1, 2, \dots, i-1\} \cup \bigcup\{\mathcal{D}(o)\colon\ o \in O(Z_{js}, D),\ D \in \mathcal{D}_{i-1},\ s \in S_{j\alpha(D)},\ j = 1, 2, \dots, i\}$.

The conditions (a)–(d) describe how to define A_{2n+1}, A_{2n+2} and $\mathcal{D}_{n+1}$. Let us suppose that there is $x_0 \in \bigcap\{A_{2n}\colon\ n \in N\}$. Then there are sequences $k_1 < k_2 < \cdots$, $(D_i)_{i=1}^\infty \in (\mathcal{D}_{k_i})_{i=1}^\infty$, $\alpha(D_1) < \alpha(D_2) < \cdots$ such that $x_0 \in \bigcap\{D_i;\ i \in N\}$. If we put $\lambda = \sup\{\alpha(D_i)\colon\ i \in N\}$ then $p_\lambda^{-1} p_\lambda(\bigcup\{F_\gamma\colon\ \gamma < \lambda\}) \cap \bigcap\{D_i\colon\ i \in N\} = \emptyset$. On the other hand, λ does not belong to S, so we have $\bigcup\{F_\gamma\colon\ \gamma < \lambda\} = F_\lambda$. From $p_\lambda^{-1} p_\lambda(F_\lambda) = X$ follows a contradiction. □

Question 6. Could we replace in Remark 2 separability by the assumption that every closed subset of X is a G_δ-set?

Remark 7. If $X \subset I^{\omega_1}$ is a hereditarily Lindelof space, $X \in \mathcal{P}$ and X is not σ-compact (without loss of generality we can assume that there is no point $x \in X$ which has a σ-compact neighborhood), then there is an Aronszajn tree $T = (T_\alpha)_{\alpha<\omega_1}$ such that for each $\alpha < \omega_1$ and $t \in T_\alpha$ there is $x_t \in X$ such that $p_\alpha(x_t) = t$, $\mathit{Int}\, p_\alpha^{-1} p_\alpha(x_t) \neq \emptyset$ and if $x \in X$ and $p_\alpha(x)$ does not belong to T_α then $p_\alpha^{-1} p_\alpha(x)$ is compact.

Question 7. What is the characterization of P-spaces of weight not greater than ω_2 belonging to $\mathcal{P}$?

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